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From Sasaki-Einstein spaces to quivers via BPS geodesics: $L^{p,q|r}$

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Abstract: The AdS/CFT correspondence between Sasaki-Einstein spaces and quiver gauge theories is studied from the perspective of massless BPS geodesics. The recently constructed toric $L^{p,q|r}$ geometries are considered: we determine the dual superconformal quivers and the spectrum of BPS mesons. The conformal anomaly is compared with the volumes of the manifolds. The $U(1)^2_F \times U(1)_R$ global symmetry quantum numbers of the mesonic operators are successfully matched with the conserved momenta of the geodesics, providing a test of AdS/CFT duality. The correspondence between BPS mesons and geodesics allows to find new precise relations between the two sides of the duality. In particular the parameters that characterize the geometry are mapped directly to the parameters used for a-maximization in the field theory.
The analysis simplifies for the special case of the $L^{p,q|q}$ models, which are shown to correspond to the known "generalized conifolds". These geometries can break conformal invariance through toric deformations of the complex structure.

Keywords: AdS-CFT Correspondence, D-branes.

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1. Introduction

This paper studies the AdS/CFT correspondence [1] in the case of four dimensional $\mathcal{N} = 1$ gauge theories. Type IIB backgrounds of the form $AdS_5 \times X^5$, where X^5 is a Sasaki-Einstein manifold, are dual to a special class of superconformal gauge theories called quivers. A quiver theory has product gauge group $\prod SU(N_i)$ and the matter fields transform in bifundamental representations. For superconformal quivers the two gravitational central charges c and a are always equal and proportional to the inverse of the volume of X^5. The moduli space of vacua is of the form $Sym_N(\mathcal{M}_3)$, where $\mathcal{M}_3$ is the Calabi-Yau cone whose base is the Sasaki-Einstein X^5. Another peculiar feature of these no-flavor theories is the natural existence of long single-trace mesonic operators, constructed from closed paths in the quivers, that are dual to semiclassical strings moving in X^5.

A special case of this is "toric AdS/CFT": the Calabi-Yau cone $\mathcal{M}_3$ is a toric manifold, admitting three $U(1)$ isometries. In other words X^5 is a T^3 fibration over a polygon, which is drawn on an integer two dimensional lattice. This small set of discrete data (that can be encoded in the $U(1)$ charges of the Gauged Linear Sigma Model fields describing the toric cone) is enough to specify completely the full geometry, and hence also the corresponding superconformal gauge theories. Of course, a more explicit description, both of the geometries and of the gauge theories, is desirable. On the geometric side, in particular, there is no known way to determine the toric Sasaki-Einstein metric on X^5 starting from the toric polygon. On the gauge side, the quiver can be thought of as a more explicit description of the algebro-geometric structure of the singularity $\mathcal{M}_3$, and an algorithm exists [3], even though it can efficiently handle only small toric polygons; see [4−8, 10, 9].

Recently various work has been done in context of toric AdS/CFT, due to the discovery of infinite sets of explicit Sasaki-Einstein metrics, in contrast to the previous knowledge of only two examples, namely S^5 and T^{11}. The latter case was analyzed by Klebanov and Witten [2]. The study of these models led to many interesting results.

[11−13] found an infinite set of Sasaki-Einstein metrics on $S^2 \times S^3$, called $\mathcal{Y}^{p,q}$. These metrics are cohomogeneity one, the isometries being $SU(2) \times U(1)^2$. One Abelian isometry, generated by the Reeb vector, is present in any, toric or not, Sasaki-Einstein manifold and is dual to the R-symmetry of the gauge theory. In [14] the toric description was found. The CY cones are quotients of $\mathbb{C}^4$; the four GLSM fields have charges $(p+q, |p-q|; -p, -p)$. The Sasaki-Einstein spaces are smooth precisely when p and q are coprime. Recently, a cohomogeneity-two generalization has been found [15, 16], leading to the so called $L^{p,q|r}$ spaces. In fact, the same local metrics on the Kähler-Einstein 4d base have been found some time ago in the mathematical literature [17]. Moreover, in [17] these metrics are shown to be the most general orthotoric Einstein metrics. The toric data of the $L^{p,q|r}$ CY cones are a simple generalization of the toric data of the $\mathcal{Y}^{p,q}$ cones. There are still only four GLSM fields and a single $U(1)$ action, with integer charges $(p, q; -r, -p-q+r)$[15]. If $p+q = 2r$ one finds $\mathcal{Y}^{\tilde{p},\tilde{q}}$.

In [18], using the toric description of the singularities, the $\mathcal{Y}^{p,q}$ superconformal quivers have been constructed; a key role was played by the $SU(2)$ global symmetry: focusing on toric quivers with this non-Abelian flavor symmetry one is basically led to the $\mathcal{Y}^{p,q}$ quivers. Various checks of the correspondence can be performed [19, 18, 20−22]. Also the marginal deformations [23] match [24, 25]. A crucial role is played by the technique of a-maximization [26], which relies on well established general properties of supersymmetric theories [27, 26] and is thus valid for any 4d SCFT. One, maybe surprising, feature of the $\mathcal{Y}^{p,q}$ theories, that is important for the present paper is the following: in the simpler Seiberg dual "phases" of the theories, namely the toric phases, there is a high degeneracy in the global symmetry quantum numbers of the bifundamental fields. This can be understood to be necessary from the AdS perspective: the smallest dibaryon operators have the same charges (modulo a factor of N), and there are only a very small number of them, since they directly correspond to the supersymmetric 3-cycles in the geometry.

We have recently witnessed a general progress in the algebro-geometric duality between toric singularities and the gauge theories living on D3 branes probing the singularities: in [28] Hanany and Kennaway put forward a correspondence between the toric data and the corresponding quivers. All toric quivers can be drawn on a torus providing a polygonalization of the torus; every superpotential term precisely corresponds to a face. The dual graph of the quiver, the dimer, has a direct physical interpretation in term of brane setups [29]. These setups significantly generalize previously known constructions [32 − 34, 8]. In [30] many features of this picture have been clarified and many examples have been given. The quiver/dimer $\leftrightarrow$ toric Calabi-Yau's correspondence is part of a general framework [31] connecting statistical mechanics of crystals and topological strings.

In [22] the periodic quiver picture was shown to encode naturally the mesonic operators of the theories. In particular, the emergence of semiclassical strings directly from paths in the periodic quivers was discussed: roughly speaking, the direction of a long path on the quiver is mapped to the position of the string in the toric base of the Sasaki-Einstein space. Massless BPS geodesic (i.e. point-like massless strings moving only along the R-charge direction) are special cases of these, and turn out to encode a great deal of information about the structure of the quiver.

The purpose of this paper is to show how the techniques of [22] can be extended to generic examples of $\mathcal{N} = 1$ AdS/CFT.

In section 2 we analyze the geometries, find the angle associated to the R-charge and determine the properties of massless point-like strings moving along this direction, that we call BPS geodesics.

Inspired by the results of [18] and [28], using the relation between the physical (p, q)-webs of five branes and toric diagrams [35, 6], we then construct the superconformal quivers associated to the $L^{p,q|r}$geometries. An important role in the determination of the global structure of the quivers is played by the mesonic BPS operators, that we determine as in [22].

The main result of the present paper is a direct comparison between these BPS mesons and BPS geodesics. As a warm up, in section 4 we discuss in detail this matching for the special cases of $L^{p,q|q}$ spaces. For these cases the gauge theories are known from [36, 34] and the analysis is simpler and somewhat more transparent, both from the gauge side and from the string side (the quartic equations in these cases become quadratic, as for the $\mathcal{Y}^{p,q}$s). The end result is a non trivial matching of the $U(1) \times U(1)$ flavor and $U(1)_R$ conserved charges between BPS geodesics and BPS mesons.

This comparison is than extended to a general $L^{p,q|r}$model in section 5. In this section we also provide a direct relation between the parameters (α, β, x_i) characterizing the manifolds [15] and the parameters used in a-maximization [26].

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2. Strings moving in the $L^{p,q|r}$ manifold

In this section we study point-like massless strings moving in the $AdS_5 \times L^{p,q|r}$ manifold whose metric is [15]:

$$ds^2 = -dt^2 \cosh^2\rho + d\rho^2 + \sinh^2\rho \, d\Omega_3^2 + ds^2_{p,q|r} \tag{2.1}$$

$$ds^2_{p,q|r} = (d\xi + \sigma)^2 + ds^2_{[4]} \tag{2.2}$$

where

$$ds^2_{[4]} = \frac{\rho^2}{4f(x)}dx^2 + \frac{\rho^2}{h(\theta)}d\theta^2 + \frac{f(x)}{\rho^2}\left(\frac{\sin^2\theta}{\alpha}d\phi + \frac{\cos^2\theta}{\beta}d\psi\right)^2 \tag{2.3}$$

$$+\frac{h(\theta)\sin^2\theta\cos^2\theta}{\rho^2}\left(\frac{(\alpha - x)}{\alpha}d\phi - \frac{(\beta - x)}{\beta}d\psi\right)^2 \tag{2.4}$$

and

$$\sigma = \frac{(\alpha - x)\sin^2\theta}{\alpha}d\phi + \frac{(\beta - x)\cos^2\theta}{\beta}d\psi \tag{2.5}$$

$$f(x) = x(\alpha - x)(\beta - x) - \mu \tag{2.6}$$

$$\rho^2 = h(\theta) - x \tag{2.7}$$

$$h(\theta) = \alpha\cos^2\theta + \beta\sin^2\theta \, . \tag{2.8}$$

The geodesics we are interested in sit at the point $\rho = 0$ of AdS_5 and move in the internal manifold. Therefore in the rest of the paper we ignore the AdS_5 part of the background.

To study such geodesics we first do a change of coordinates and then properly identify the angle conjugated to the R-charge. With that information we find the relation between the conserved charges for some particular cases of interest which we call extremal geodesics.

2.1 A change of coordinates

In order to facilitate the comparison between BPS geodesics and BPS mesons in the gauge theory we redefine the variables by $y = \cos(2\theta)$, getting

$$ds^2_{[5]} = (d\xi + \sigma)^2 + ds^2_{[4]} \tag{2.9}$$

with

$$\sigma = \frac{(\alpha - x)(1 - y)}{2\alpha}d\phi + \frac{(\beta - x)(1 + y)}{2\beta}d\psi \, . \tag{2.10}$$

In such a way the two functions σ_i in $d\xi + \sigma_\phi d\phi + \sigma_\psi d\psi$ are products of two linear functions of one coordinate on the toric base (x and y). This fact arises naturally when comparing with the gauge theory.

The local Kähler-Einstein metric takes the fairly symmetric form

$$ds^2_{[4]} = \frac{\rho^2}{4f(x)}dx^2 + \frac{f(x)}{4\rho^2}\left(\frac{(1 - y)}{\alpha}d\phi + \frac{(1 + y)}{\beta}d\psi\right)^2 + \tag{2.11}$$

$$+\frac{\rho^2}{4g(y)}dy^2 + \frac{g(y)}{4\rho^2}\left(\frac{(\alpha - x)}{\alpha}d\phi - \frac{(\beta - x)}{\beta}d\psi\right)^2 \tag{2.12}$$

with

$$f(x) = x(\alpha - x)(\beta - x) - \mu \tag{2.13}$$

$$g(y) = \frac{1}{2}(\alpha + \beta - y(\alpha - \beta))(1 - y^2) \tag{2.14}$$

$$\rho^2 = \frac{1}{2}(\alpha + \beta - y(\alpha - \beta) - 2x)\,. \tag{2.15}$$

With $\alpha > \beta$ the ranges of the coordinates on the base is $x_1 \leq x \leq x_2$ and $-1 \leq y \leq 1$, where $0 \leq x_1 \leq x_2 \leq x_3$ are the three roots of $f(x)$. Since $x \leq x_2 \leq \beta$ we have $\rho^2 \geq \frac{1}{2}(\alpha - \beta)(1 - y) \geq 0$, for $y \leq 1$. Note also that $g(y)$ is a cubic function of y as $f(x)$ is of x.

2.2 R-charge

To be able to compare the results for massless strings with the field theory side we have to identify the angle conjugate to the R-symmetry. In order to do that, we need to discuss briefly the computation of the holomorphic 3-form on the Calabi-Yau cone $\mathcal{M}_3^{p,q|r}$. This is because the holomorphic three form can be written as $\Omega_{ijk} = \eta^T \Gamma_{ijk} \eta$ with η the covariantly constant spinor. The R-charge rotates the covariantly constant spinor as $\eta \rightarrow e^{\frac{1}{2} i \alpha} \eta$ and $\Omega \rightarrow e^{i\alpha} \Omega$.

With the 1-form σ defined in the four dimensional base of the Sasaki-Einstein manifold, we compute its Kähler form k and complex structure J as:

$$k = -\frac{1}{2} d\sigma, \qquad J_a^b = k_{ac} g^{cb}, \qquad P_a^b = \frac{1}{2}\left(\delta_a^b + i J_a^b\right) \tag{2.16}$$

where P_a^b projects out the anti holomorphic components. With this projector, we can find two holomorphic 1-forms $\eta_{1,2}$:

$$\eta_1 = \frac{x - \beta}{f(x)} dx - \frac{1 + y}{g(y)} dy + \frac{2i}{\alpha} d\phi \tag{2.17}$$

$$\eta_2 = \frac{x - \alpha}{f(x)} dx + \frac{1 - y}{g(y)} dy + \frac{2i}{\beta} d\psi\,. \tag{2.18}$$

The Sasaki-Einstein metric allows to construct a Calabi-Yau cone $\mathcal{M}_3^{p,q|r}$ with metric

$$ds^2 = dr^2 + r^2 ds^2_{[5]}\,. \tag{2.19}$$

In this manifold we introduce a third holomorphic form

$$\eta_3 = \frac{dr}{r} + i\,(d\xi + \sigma)\,. \tag{2.20}$$

Now, as argued by Martelli and Sparks in [16], the covariantly constant holomorphic 3-form follows as

$$\Omega_{[3]} = \sqrt{f(x) g(y)}\; e^{i\psi_R}\; r^3\; \eta_1 \wedge \eta_2 \wedge \eta_3\,. \tag{2.21}$$

The phase ψ_R is conjugated to the R-charge and can be determined to be

$$\psi_R = 3\xi + \phi + \psi\,. \tag{2.22}$$

from the condition that $\Omega_{[3]}$ should be covariantly constant.

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We can therefore rewrite the 5d metric as

$$ds^2_{[5]} = \left(\frac{d\psi_R}{3} + \left(\frac{(\alpha - x)(1-y)}{2\alpha} - \frac{1}{3} \right) d\phi + \left(\frac{(\beta - x)(1+y)}{2\beta} - \frac{1}{3} \right) d\psi \right)^2 + ds^2_{[4]} \,. \quad (2.23)$$

2.3 BPS geodesics

The action for a massless particle moving along the internal manifold can be written as

$$S = \frac{1}{2} \left\{ (\frac{1}{3}\dot\psi_R + a_1\dot\phi + a_2\dot\psi)^2 + b_1\dot x^2 + b_2\dot y^2 + (c_1\dot\phi + c_2\dot\psi)^2 + (d_1\dot\phi - d_2\dot\psi)^2 \right\} \quad (2.24)$$

with the definitions

$$a_1 = \frac{(\alpha - x)(1-y)}{2\alpha} - \frac{1}{3}, \quad a_2 = \frac{(\beta - x)(1+y)}{2\beta} - \frac{1}{3}, \quad b_1 = \frac{\rho^2}{4f(x)}, \quad b_2 = \frac{\rho^2}{4g(y)} \quad (2.25)$$

$$c_1 = \frac{\sqrt{f(x)}}{2\rho}\frac{(1-y)}{\alpha}, \quad c_2 = \frac{\sqrt{f(x)}}{2\rho}\frac{(1+y)}{\beta}, \quad (2.26)$$

$$d_1 = \frac{\sqrt{g(y)}}{2\rho}\frac{(\alpha - x)}{\alpha}, \quad d_2 = \frac{\sqrt{g(y)}}{2\rho}\frac{(\beta - x)}{\beta}\,. \quad (2.27)$$

We can compute the conjugate momenta:

$$P_{\psi_R} = \frac{1}{3}\left(\frac{1}{3}\dot\psi_R + a_1\dot\phi + a_2\dot\psi\right) \quad (2.28)$$

$$P_y = b_2\dot y \quad (2.29)$$

$$P_x = b_1\dot x \quad (2.30)$$

$$P_\phi = 3a_1 P_{\psi_R} + c_1(c_1\dot\phi + c_2\dot\psi) + d_1(d_1\dot\phi - d_2\dot\psi) \quad (2.31)$$

$$P_\psi = 3a_2 P_{\psi_R} + c_2(c_1\dot\phi + c_2\dot\psi) - d_2(d_1\dot\phi - d_2\dot\psi) \quad (2.32)$$

and the Hamiltonian

$$H = \frac{9}{2}P^2_{\psi_R} + \frac{1}{2b_2}P_y^2 + \frac{1}{2b_1}P_x^2 + \frac{1}{2}(\sigma_1^2 + \sigma_2^2) \quad (2.33)$$

where

$$\sigma_1 = (c_1\dot\phi + c_2\dot\psi) \quad (2.34)$$

$$\sigma_2 = (d_1\dot\phi - d_2\dot\psi)\,. \quad (2.35)$$

In terms of the momenta we get

$$\sigma_1 = -\frac{-d_2P_\phi - d_1P_\psi + 3(a_1d_2 + a_2d_1)P_{\psi_R}}{c_1d_2 + c_2d_1} \quad (2.36)$$

$$\sigma_2 = -\frac{-c_2P_\phi + c_1P_\psi + 3(a_1c_2 - a_2c_1)P_{\psi_R}}{c_1d_2 + c_2d_1}\,. \quad (2.37)$$

Now we consider geodesics that satisfy $P_y = 0$, $P_x = 0$ implying that $x = x_0$ and $y = y_0$ with x_0 and y_0 constant. These constants should be chosen so as to minimize H as follows

	y	x	P_ϕ/P_{ψ_R}	P_ψ/P_{ψ_R}
LD	-1	x_1	$3\left(\frac{2}{3}-\frac{x_1}{\alpha}\right)$	-1
RU	$+1$	x_2	-1	$3\left(\frac{2}{3}-\frac{x_2}{\beta}\right)$
LU	$+1$	x_1	-1	$3\left(\frac{2}{3}-\frac{x_1}{\beta}\right)$
RD	-1	x_2	$3\left(\frac{2}{3}-\frac{x_1}{\alpha}\right)$	-1

Table 1: Values of the flavor charges for the four extremal BPS geodesics. The names are related to their field theory interpretation.

form the eq. of motion for x and y. The minimum is when $\sigma_1 = \sigma_2 = 0$, which we expect to correspond to a BPS geodesic.

This implies $\dot{\phi} = \dot{\psi} = 0$ and from (2.32):

$$P_\phi = 3a_1 P_{\psi_R}, \qquad P_\psi = 3a_2 P_{\psi_R}\,. \tag{2.38}$$

Using the definitions (2.25) we can rewrite this as

$$\frac{P_\phi}{P_{\psi_R}} = \frac{\alpha - 3\alpha y - 3x + 3xy}{2\alpha} \tag{2.39}$$

$$\frac{P_\psi}{P_{\psi_R}} = \frac{\beta + 3\beta y - 3x - 3xy}{2\beta}\,. \tag{2.40}$$

These eqs. give the ratios of the conserved charges as a function of the (x, y) position of the BPS geodesic. We will be in particular interested in the values of these ratios at the four vertices of the rectangle parameterized by (x, y). Here the conserved momenta assume special values that we show in table 1. Since multiplying two operators the $U(1)$ charges simply add, it is enough to check the duality between geodesics and BPS mesons at these four extremal points, that corresponds to the four vertices of the toric diagrams, as we see in the next section.

3. From GLSM charges to quivers and chiral rings: $L^{p,q|r}$

In this section we pass from the toric data of the $L^{p,q|r}$singularities to a representation of the gauge theories.[1] Modulo redefinition of the variables, one can assume $p \leq s \leq r \leq q$ (recall $p + q = r + s$). We also assume that there is no overall common divisor for the p, q, r, s, otherwise one is dealing with Abelian orbifolds.

Since the coordinates on the toric base lie on a polygon with four edges, it is clear that the toric diagram have four edges as well.[2] Using $SL(2, \mathbb{Z})$ redefinitions of the toric quadrangle, one can assume that two vertices are in position $(0, 0)$ and $(1, 0)$. A generic four-edges toric diagram is reported in figure 1. The natural correspondence between toric

[1] This problem is addressed from the dimer perspective of [28, 30] also in [37, 38] and [39]

[2] As we will see from the BPS mesons, if some of the four pairs (p, s), (p, r), (q, s), (q, r) have a non trivial common divisor, there are additional points on the boundary of the toric diagram. For the moment we will assume no non trivial common divisors, that leads [16] to smooth geometries.

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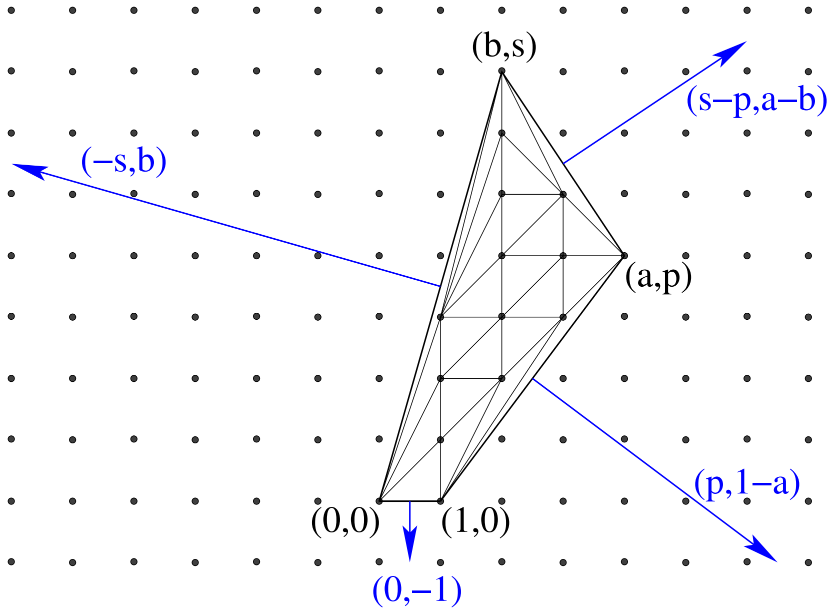


Figure 1: Generic toric diagram with four edges. The related (p, q)-web is depicted.

diagrams and (p, q)-webs of five branes [35] is also emphasized. As explained in [6], going into the mirror picture, one can interpret the number of bifundamental fields of the quiver as intersection numbers of 3-cycles in the mirror picture. These intersection numbers correspond to "intersections" of (p, q)-branes, given by the vector product [6]:

$$(\mathcal{C}_i \circ \mathcal{C}_j) = p_i q_j - p_j q_i \,. \tag{3.1}$$

Using this simple formula we compute the (p, q)-branes intersection for our case of interest in figure 2.

Since intersection numbers can be thought as vector products, they are positive when the sine of the angle between vectors is positive. Since the multiplicities are positive that allows to determine the direction of the arrows denoting the bifundamental fields. It is easy to see that the multiplicities depend only on the combination $q = as - pb$ (using $SL(2, \mathbb{Z})$ redefinitions, one can assume $a \leq p$). It is also convenient to define $r = p + q - s$ (implying $p + q = r + s$). The resulting bifundamentals together with their multiplicities are depicted in figure 3. With our definitions, we are thus describing the toric diagram corresponding to $GLSM$ charges $(p, q; -r, -p - q + r)$ [15]. p, q, r, s are precisely the multiplicities of four of the type of the fields that have to appear in the toric phases of the quiver. According to their direction, we call these four fields $U \uparrow$, $D \downarrow$, $L \leftarrow$ and $R \rightarrow$.

Figure 3 should be thought of as the "folded" superconformal quiver. In other words one can read off form that the number of the various types of fields and the number of nodes. There are two additional diagonal fields $\nwarrow$ and $\swarrow$.[3] This representation is very

[3]The picture we are proposing here is expected to be valid only for a subset of the toric phases. For examples for the $\mathcal{Y}^{p,q}$ in the so called "double impurity phases" there are also diagonal fields pointing rightward [40].

Field	Number	R_0	Q_H	Q_V	Q_3	Q_B
R $\rightarrow$	q	1	$+1$	0	0	+ p
L $\leftarrow$	p	1	-1	0	$+1$	+ q
U $\uparrow$	s	0	0	$+1$	-1	$-r$
D $\downarrow$	r	0	0	-1	0	$-s$
$\nwarrow$	q $-$ s	1	-1	$+1$	0	q $-$ r
$\swarrow$	q $-$ r	1	-1	-1	$+1$	q $-$ s

Table 2: Charge assignments for the basic fields. The charge Q_3 is redundant but plays a role in the calculations. R_0 is a trial R-charge which satisfies the same anomaly cancellation constraints as the actual R-charge.

useful in determining the global symmetry quantum numbers of the bifundamentals. We call the two toric $U(1)$ symmetries J_H and J_V, according to their orientation. The values are reported in table 2.

Another way of finding the global symmetries from figure 3 is to associate a symmetry to every of the four vertices, say $Q_1, \ldots, Q_4$. The charges are $+1$ for bifundamentals outgoing from that vertex, -1 for ingoing bifundamentals and 0 for the others. The sum of these four symmetries vanishes, and a particular linear combination can be seen to be the baryonic symmetry.[4]

The way to prove that all the field with the same direction have the same global symmetry quantum number (recall however that they have different gauge quantum numbers) is simply to impose the vanishing of the anomalies of these currents for each node.

By studying the possible terms in the superpotential one concludes that the toric representation of the quiver can be constructed using the blocks depicted in figure 5. These block should be glued, respecting the orientation of the sides, into a fundamental domain with a given number of each type. Finally it should be possible to draw the fundamental domain on a torus (equivalently it should be possible to tile the plane with it) giving rise to identifications between the points in the boundary of such domain. This is better described with an example. In figure 6 and table 7 we give a construction with $n_a = 1$, $n_b = 3$, $n_c = 1$.

In general, the number of blocks of type (a), (b) and (c) are: $n_a = p$, $n_b = q - s$, $n_c = q - r$. The total number of nodes in the quiver is $p + q$, the total number of bifundamental fields is $p + 3q$. This gives a total number of each type of field in agreement with table 2.

Gluing these fundamental blocks gives rise to vertices of the type in figure 4. The field theory interpretation is that each vertex corresponds to an $SU(N)$ gauge group and therefore implies anomaly cancellation conditions for the fields ending or emerging from it, or equivalently the beta function of the corresponding coupling should be zero.

Also, in the blocks of figure 5, each square or triangular face corresponds to a term in the superpotential and therefore also implies a relation between the R-charges of the field

[4] In general the number of baryonic symmetries is equal to $n - 3$, where n is the number of external (p, q)-branes. In our picture, for more generic toric diagrams, one constructs immediately n global Abelian symmetries. The sum is always decoupled, two of them are the standard toric Abelian isometries, and the remaining $(n - 3)$ are the baryonic symmetries.

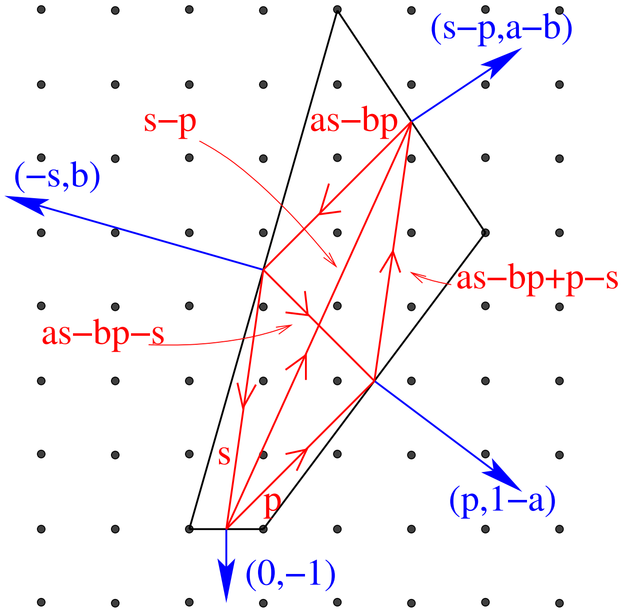


Figure 2: Generic toric diagram with four edges. The intersection numbers are computed.

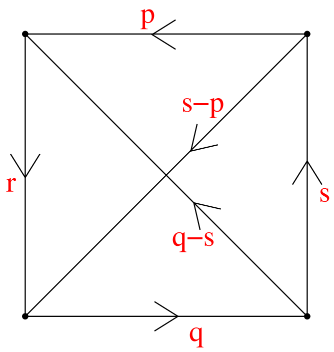


Figure 3: From figure 2 we extract the multiplicities of the bi-fundamental fields. We define $q = as - bp$ and $r = p + q - s$.

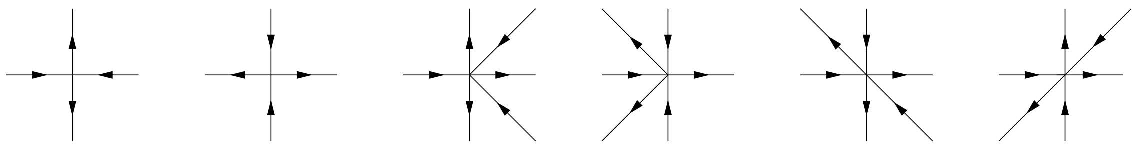


Figure 4: Vertices (or nodes) appearing in the toric representation of the quiver.

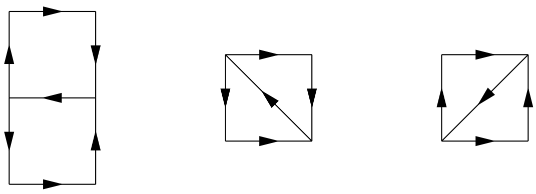
surrounding it. Namely, they have to add up to two if the corresponding superpotential coupling has zero beta function.

The constraints reduce to the following independent equations

$$R_{\rightarrow} + R_{\leftarrow} + R_{\uparrow} + R_{\downarrow} = 2 \tag{3.2}$$

$$R_{\rightarrow} + R_{\downarrow} + R_{\nwarrow} = 2 \tag{3.3}$$

$$R_{\uparrow} + R_{\swarrow} + R_{\rightarrow} = 2\,. \tag{3.4}$$



a b c

Figure 5: Building blocks of the toric representation of the quiver.

A simple way to satisfy all this constraints is to assign the following R-charges $R_{\uparrow} = R_{\downarrow} = 0$, $R_{\leftarrow} = R_{\rightarrow} = R_{\nwarrow} = R_{\swarrow} = 1$.

However we can add an arbitrary solution of the homogeneous equations which can be parameterized in terms of three real numbers $(x,\, y,\, z)$:

$$
\begin{aligned}
R_{\rightarrow} &= 1 + x \\
R_{\leftarrow} &= 1 - x + z \\
R_{\uparrow} &= y - z \\
R_{\downarrow} &= -y \\
R_{\nwarrow} &= 1 - x + y \\
R_{\swarrow} &= 1 - x - y + z\,.
\end{aligned}
\tag{3.5}
$$

It is easily seen that x and y correspond to Q_H and Q_V, the $U(1) \times U(1)$ global symmetries of the theory.

On the other hand z can be associated with a $U(1)$ symmetry Q_3 in terms of which the baryonic charge is written as $Q_B = pQ_H + sQ_V + (p+q)Q_3$. The charge assignments are summarized in table 2. There are only three independent charges. One can use Q_3 which is simpler or the baryonic charge which conveys more physical information since under it all meson operators are neutral.

The global symmetry currents satisfy $tr(R_0) = tr(J_B) = tr(J_H) = tr(J_V) = 0$. This is due to the quiver structure of the theory:

$$
tr(J) = \sum_{f \in fields} J[f] = \frac{1}{2} \sum_{i,j \in nodes} J[f_{i,j}] = 0\,. \tag{3.6}
$$

The last term vanish because, for each i, $\sum_j J[f_{i,j}] = 0$, due to the fact that the symmetries are free of anomalies.[5]

One can also check that $tr(J_B^3) = 0$, as has to be the case for a baryonic symmetry. In our parameterization, the two $U(1)$ symmetries, Q_H and Q_V, also happen to satisfy $tr(J_F^3) = 0$. This fact changes by mixing the two flavor currents between each other or

[5]In the case of toric quivers, for the non-R currents, one can see this by noticing the $tr(J)$ can be recast as (one half) the sum over all the faces of the total charges of the face, which has to vanish since the superpotential respects the symmetry. For the R-symmetry, a similar condition tells that the quiver lives on a two-torus [30].

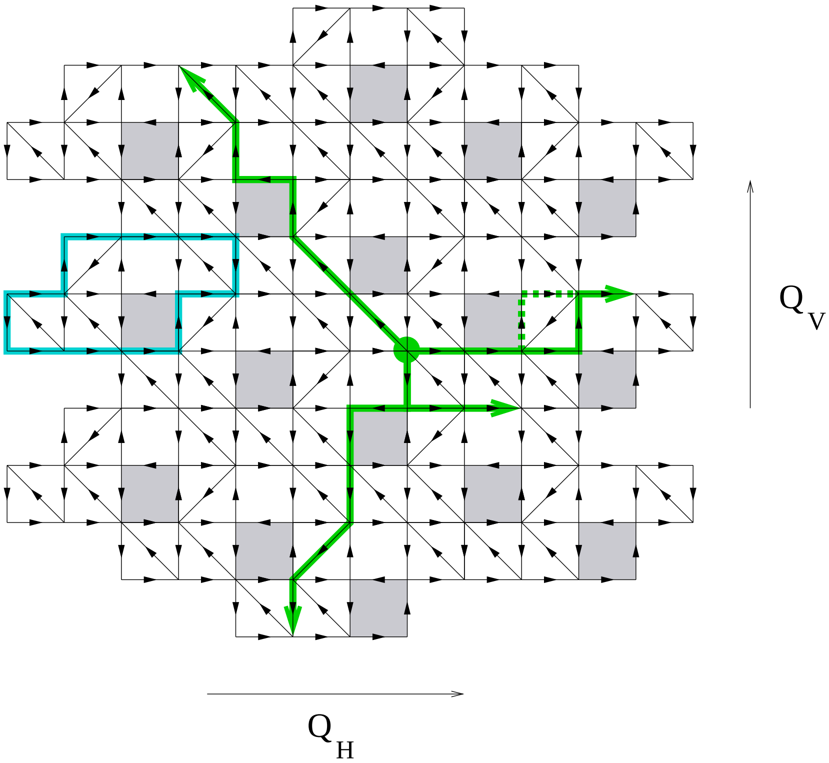


Figure 6: Periodic quiver for the $L^{1,5|4}$ model. The four "extremal" BPS meson and the fundamental domain are highlighted. One square is shaded to make the knight periodicity easier to appreciate. The dotted line shows an alternative LU (left-up) operator with the same charges. The operator in the chiral ring is a linear superposition of the various possibilities.

with the baryonic current. The identification of J_B with the baryonic symmetry comes from the fact, as we will see, that all mesonic operators (traces of products of bifundamentals) are uncharged under this symmetry.

3.1 BPS mesons

In this subsection we want to determine the BPS mesonic operators of the theories, that will be compared with the massless BPS geodesics. The BPS operators that we need to compare with the geodesic are such that they have maximal $U(1)$ charges (in modulus) for given R-charge. These are the geodesics that lie precisely on the boundary of the coordinate rectangle. There are four boundaries, corresponding to the 4 supersymmetric 3-cycles, and four vertices. The operators that correspond to these four "corner" geodesics thus encodes all the information about BPS geodesics. As a consequence we focus on these four extremal mesons. It is clear also that these four mesons precisely correspond to the four (p, q)-branes. The study of the mesons goes further with respect to the folded quiver picture of the last subsection, and needs a precise understanding of the way in which the blocks are glued together. If there is an overall common factor for the four GLSM charges $p, q, -r, -s$, moreover, there can be various quivers corresponding to the same folded quiver. The simplest examples are the two different $\mathbb{Z}_2$ quotients of the conifold.

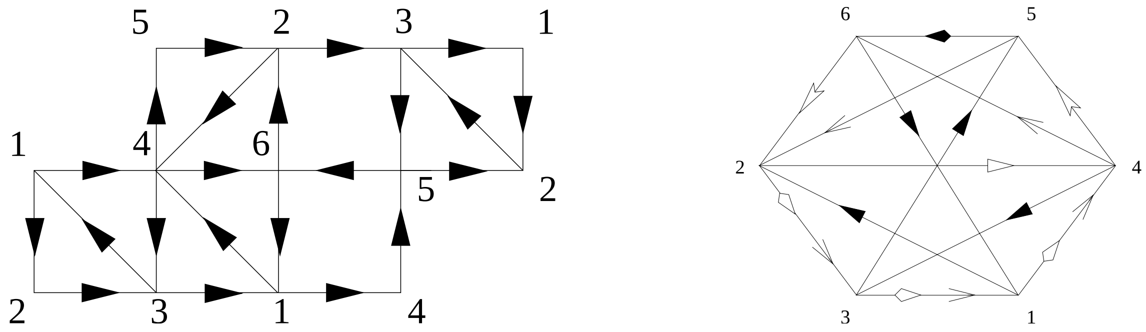


Figure 7: The fundamental domain of $L^{1,5|4}$ and the traditional quiver. The numbers identify the six gauge groups and also determine how the fundamental domain is tiled to cover the plane as in figure 6. Notice that whereas the quiver only contains the information about the matter content of the theory, the periodic quiver, also tell the superpotential [28, 30].

Meson	R_0	R	Q_H	Q_V	Q_B
$\mathcal{O}_{LD}$	s	$-qy+s(1-x+z)$	$-s$	$-q$	0
$\mathcal{O}_{RU}$	r	$r(1+x)+p(y-z)$	$+r$	$+p$	0
$\mathcal{O}_{LU}$	r	$r(1-x+z)+q(y-z)$	$-r$	$+q$	0
$\mathcal{O}_{RD}$	s	$s(1+x)-py$	$+s$	$-p$	0

Table 3: Charge assignments for the four extremal BPS mesons. The variables (x,y,z) are taken at the local maximum of the central charge a.

Assuming no overall common divisor there is only one quiver; determining this gives also the global charges of our four extremal BPS operators.

The extremal operator going in the right-up direction, $\mathcal{O}_{RU}$, for instance, is composed of $\rightarrow$ and $\uparrow$ fields. The extremal operator going in the left-up direction, $\mathcal{O}_{LU}$ is composed only of $\leftarrow$, $\uparrow$ and $\nwarrow$ fields. Similarly for the other two operators. These requirements uniquely determine the four operators.

Without entering into the details, the end result is pretty simple. We assume for the moment that $\gcd(p,s) = \gcd(p,r) = \gcd(q,s) = \gcd(q,r) = 1$ and summarize the general results in table 3. As an explicit example, in figure 6 we depict the four extremal operators for the particular case of $L^{1,5|4}$.

We can now proceed to understand what happens when there is a non trivial common divisor between p or q and r or s. In these cases it is easy to see that the toric diagrams, determined by the construction of the previous subsection in term of the GLSM fields, have additional points on the edges. This corresponds to additional (p,q)-legs and reflects as non trivial multiplicities for our four BPS mesons. Namely there are $\gcd(s,q)$ $\mathcal{O}_{LD}$ operators, $\gcd(r,p)$ $\mathcal{O}_{RD}$ operators, and so on. Correspondingly, these chiral ring generators become shorter, and their $U(1)$ charges are divided by $\gcd(s,q)$ for $\mathcal{O}_{LD}$, by $\gcd(r,p)$ for $\mathcal{O}_{RD}$ etc. This does not affect the ratio between charges. So, for the purpose of comparing with the geodesics, this fact does play a relevant role.

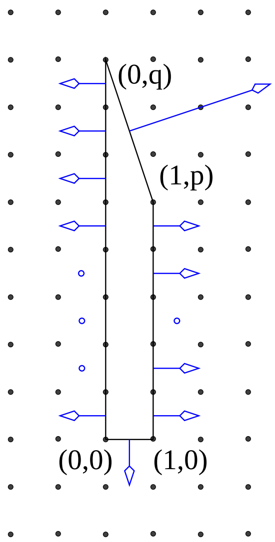


Figure 8: Toric diagrams inside the strip.

4. Resolving the strip: $L^{p,q|q}$

In this section we focus on a subset of the $L^{p,q|r}$ models that lies at the opposite boundary with respect to the $Y^{p,q}$ models, namely $L^{p,q|q}$. In these case one gets the so called generalized conifolds, studied in detail in [36, 34]. The corresponding toric diagrams have no internal points and lie inside the so called strip.

We restrict as usual to $p \leq q$. For $p = q$ we have $\mathbb{Z}_q$ orbifolds of the conifold, and for $p = 0$ the $\mathcal{N} = 2$ Abelian $\mathbb{Z}_q$ orbifolds of S^5. The $Y^{p,q}$ models can be thought of as an interpolation between S^5 and T^{11} that preserves an $U(2)$ flavor symmetry [20]. In the same way the $L^{p,q|q}$ models can be thought of as an interpolation between S^5 and T^{11} that preserves an $U(1)^2$ flavor symmetry and complete non-chirality. In these cases, instead of having six types of fields as for a general $L^{p,q|r}$, there are only five types, with multiplicities p, q, $q-p$. The charges are given in the table 4. As it can be seen from the toric diagrams, there are parallel external (p,q)-legs. This implies (see [43, 21, 44] for recent discussions) that there are *toric* complex structure deformations of these models, as is the case of the conifold [42], which is also $L^{1,1|1}$. This corresponds to the possibility of pulling out one or more branes from the (p,q)-web.

In addition to the two toric flavor symmetries, there are $p+q-1$ baryonic symmetries. These symmetries, together with the combination $J_H + J_V$ of flavor symmetries, are not to be taken into consideration in performing a-maximization. The reason is that the quivers we are considering are completely non chiral. This means that the bifundamentals are either adjoint fields or come in pairs. Every pair contains two bifundamentals with opposite gauge quantum numbers. Since also the interactions are non chiral, it is clear that the two bifundamentals of every pair have they same r-charge. We can thus impose this equality before doing the maximization. In our case we see that L-fields are in the same pair of the U-fields. The other pair is D- and R-fields. One is thus left to maximize a one parameter family of symmetries, that can be taken to be $R_0 + \lambda(J_H - J_V)$. The results

Field	number	Q_R	Q_H	Q_V
L	p	$\frac{2q-p-\sqrt{p^2-pq+q^2}}{3(q-p)}$	-1	0
R	q	$\frac{q-2p+\sqrt{p^2-pq+q^2}}{3(q-p)}$	$+1$	0
U	p	$\frac{2q-p-\sqrt{p^2-pq+q^2}}{3(q-p)}$	0	$+1$
D	q	$\frac{q-2p+\sqrt{p^2-pq+q^2}}{3(q-p)}$	0	-1
A	$q-p$	$\frac{4q-2p-2\sqrt{p^2-pq+q^2}}{3(q-p)}$	-1	$+1$
$\mathcal{O}_{LD}$	1	$\frac{q+p+\sqrt{p^2-pq+q^2}}{3}$	$-p$	$-q$
$\mathcal{O}_{RU}$	1	$\frac{q+p+\sqrt{p^2-pq+q^2}}{3}$	$+q$	$+p$
$\mathcal{O}_{LU}$	q	$\frac{4q-2p-2\sqrt{p^2-pq+q^2}}{3(q-p)}$	-1	$+1$
$\mathcal{O}_{RD}$	p	$\frac{2q-4p+2\sqrt{p^2-pq+q^2}}{3(q-p)}$	$+1$	-1

Table 4: Charges of the fields for the $L^{p,q|q}$ quivers.

have thus to be quadratic irrational R-charges, even if in this case there is no non-Abelian symmetry.

The final results for the R-charges is given in table 4. The central charges is, from the formulas of [27]

$$c = a = \frac{9}{32} tr(R^3) = \frac{2p^3 - 3p^2 q - 3pq^2 + 2q^3 + 2\sqrt{(p^2 - pq + q^2)^3}}{16(q-p)^2} . \tag{4.1}$$

The AdS/CFT formula [1, 41] $a = \frac{\pi^3}{4V}$ relates this central charge to the volume of the $L^{p,q|q}$ manifold. The result for the volumes is given in [15] in terms of the solutions of a quartic equation. It is easy to see that such equation becomes quadratic in the case $p = s$, $q = r$ and the positive solution matches precisely the field theory result.

It is important that the results obtained so far in this section can be applied to different quivers, if q and p are not coprime integers. This corresponds to the possibility of taking different Z_k orbifolds of the same toric Sasaki-Einstein manifold. In this case the chiral ring generators can be different from the ones we discuss. We thus restrict to q and p coprime. In this case there is only one quiver, modulo Seiberg dualities. The chiral ring is generated by four different types of operators. One subtlety is that some of these operators come with non trivial multiplicities, this is due to the non smoothness of the background. From the (p, q)-web of 5 branes (corresponding directly to the toric diagrams) this fact can be seen as the presence of parallel external legs. It is the analog of the presence of two $\mathcal{L}_-$ operators with spin 0 in $Y^{p,p}$ [22].

There are two long operators, one, $\mathcal{O}_{LD}$, made of p L-fields and q D–fields and one, $\mathcal{O}_{RU}$, made of p U-fields and q R-fields. There are $q-p$ length 1 operators made with the $q-p$ different adjoint A-fields, that come together with the p mesons of the form $tr(UL)$. Finally, we find p length 2 of the form $tr(DR)$.

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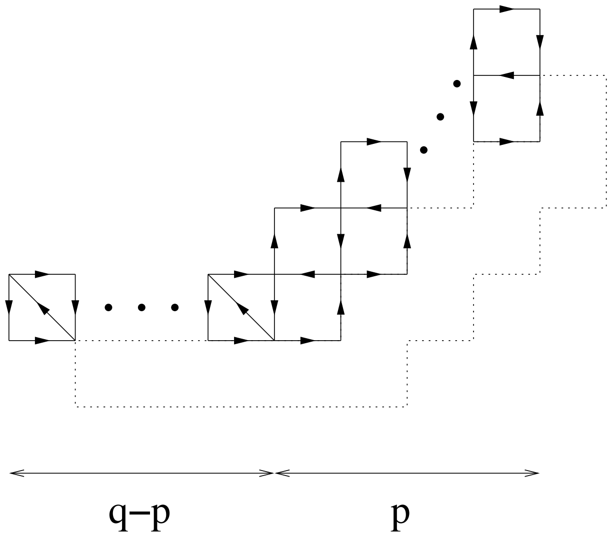

Figure 9: A toric phase for the generalized conifold. We show in the dotted lines how to glue the fundamental domain.

Meson	Q_H	Q_V	Q_ϕ/Q_R	Q_ψ/Q_R
$\mathcal{O}_{LD}$	$-p$	$-q$	-1	$\frac{p+2q-2\sqrt{p^2-pq+q^2}}{p}$
$\mathcal{O}_{RU}$	$+q$	$+p$	$\frac{2p+q-2\sqrt{p^2-pq+q^2}}{q}$	-1
$\mathcal{O}_{LU}$	-1	$+1$	$\frac{-p+q+\sqrt{p^2-pq+q^2}}{q}$	-1
$\mathcal{O}_{RD}$	$+1$	-1	-1	$\frac{p-q+\sqrt{p^2-pq+q^2}}{p}$

Table 5: Charge assignments for the four extremal mesonic fields.

Now we redefine the two flavor charges Q_H and Q_V in such a way that for each meson one of the two new charges, that we call Q_ϕ and Q_ψ, satisfy $Q_\phi = -Q_R$ or $Q_\psi = -Q_R$. We display the charges in table 5. This gauge theoretical table is meant to be compared with the geometrical table 1 of section 2. By construction the (-1)s in the last two columns match. We now compute the other four values from the geometry.

4.1 Comparison with the geometry

In the case of $r = q$ the geometrical formulas of the Appendix simplify. eqs. (7.9) and (7.8) become

$$0 = p - 4q\chi_1 - 4(p-q)\chi_1^2 \tag{4.2}$$

$$\chi_1 + \chi_2 = \frac{1}{2} \tag{4.3}$$

which give

$$\chi_1 = \frac{-p + \sqrt{p^2 - pq + q^2}}{2(q-p)} \tag{4.4}$$

$$\chi_2 = \frac{q - \sqrt{p^2 - pq + q^2}}{2(q-p)}\,. \tag{4.5}$$

Due to eq. (4.3), the square root in eq. (7.10) simplifies to

$$\sqrt{1-2(\chi_1+\chi_2)+(\chi_2-\chi_1)^2}=\chi_2-\chi_1\geq 0\,. \tag{4.6}$$

We thus find

$$\frac{2}{3}-\frac{x_1}{\alpha}=\frac{p+2q-2\sqrt{p^2-pq+q^2}}{3p} \tag{4.7}$$

$$\frac{2}{3}-\frac{x_2}{\alpha}=\frac{p-q+\sqrt{p^2-pq+q^2}}{3p} \tag{4.8}$$

$$\frac{2}{3}-\frac{x_1}{\beta}=\frac{q-p+\sqrt{p^2-pq+q^2}}{3q} \tag{4.9}$$

$$\frac{2}{3}-\frac{x_2}{\beta}=\frac{q+2p-2\sqrt{p^2-pq+q^2}}{3q}\,. \tag{4.10}$$

These are precisely the values found on the gauge side, reported in table 5.

5. General case: $L^{p,q|r}$

5.1 R-charges

In this section we perform a-maximization [26] to obtain the R-charges and compare to the previous results. As a check we also compare the central charge we obtain to the volume of the Sasaki-Einstein manifold.

Given the charge assignments of table 2 and eq. (3.5), the a-function that we should maximize can be written as

$$\begin{aligned}\mathrm{tr}((R-1)^3)=\frac{32}{9}a&=p+q+qx^3+p(z-x)^3+s(y-z-1)^3\\&\quad+r(-y-1)^3+(q-s)(y-x)^3+(q-r)(z-x-y)^3\end{aligned} \tag{5.1}$$

where we included the contribution $p+q$ from the gauginos. Now we have to find the point at which $\partial_x a=\partial_y a=\partial_z a=0$. It is useful to introduce two new variables $\xi_1=x/z$ and $\xi_2=y/z$. The first equations to solve implies

$$\frac{\partial a}{\partial x}=0 \qquad\Longleftrightarrow\qquad \xi_1=\frac{1}{2}\frac{s+2(r-q)\xi_2+(q-p)\xi_2^2}{s+(r-s)\xi_2}\,. \tag{5.2}$$

Then we get

$$s\frac{\partial a}{\partial y}+(s-r)\frac{\partial a}{\partial z}=0 \qquad\Longleftrightarrow \tag{5.3}$$

$$z=\frac{2rs(1-2\xi_2)}{p(s-r)\xi_2^2+2\xi_1\xi_2(q(r+s)-s^2-r^2)+2pr\xi_2+2s(p-r)\xi_1-sp}\,. \tag{5.4}$$

Finally, replacing the expressions for z and ξ_1 in the equation $\frac{\partial a}{\partial z}=0$ we get a quartic equation for ξ_2:

$$P_{[4]}(\xi_2)=0 \tag{5.5}$$

where $P_{[4]}$ is a polynomial of order four given by

$$
\begin{aligned}
P_{[4]}(\xi_2) = & \ 4(-4r_-^2 p_+^2 + p_+^2 p_-^2 + 3r_-^4)\ \xi_2^4 \\
& +(32r_-^2 p_+^2 + 4r_- p_+ p_-^2 + 12r_-^3 p_+ - 24r_-^4 - 8p_+^2 p_-^2 - 16r_- p_+^3)\ \xi_2^3 \\
& +(r_-^2 p_-^2 + 19r_-^4 - 21r_-^2 p_+^2 - 6r_- p_+ p_-^2 - 18r_-^3 p_+ - 4p_+^4 + 5p_+^2 p_-^2 + 24r_- p_+^3)\ \xi_2^2 \\
& +(10r_-^3 p_+ - r_-^2 p_-^2 - p_+^2 p_-^2 - 7r_-^4 - 12r_- p_+^3 + 5r_-^2 p_+^2 + 4p_+^4 + 2r_- p_+ p_-^2)\ \xi_2 \\
& +r_-^4 - p_+^4 + 2r_- p_+^3 - 2r_-^3 p_+
\end{aligned}
\tag{5.6}
$$

where, for brevity we defined $p_\pm = p \pm q$ and $r_- = r - s$. When $r = s$ or $p = r$ corresponding to $r_- = 0$ or $r_- = p_-$, the equation factorizes. In the $r_- = 0$ case, which corresponds to $Y^{\frac{1}{2}p_+, -\frac{1}{2}p_-}$, a solution is $\xi_2 = \frac{1}{2}$ which can be seen to agree with the known result.

5.2 The volume of the manifold

A way to check the a-maximization we performed is to compute a and compare with the volume of the $L^{p,q|r}$ manifold. The value of a at the local maximum can be seen to be[6]

$$
\frac{32}{9}\ \bar{a} = r + s - r(1+\bar{y})^2 - s(1 - \bar{y} + \bar{z})^2 \tag{5.7}
$$

where the bars indicate quantities evaluated at the local maximum. We can now obtain an expression in terms of ξ_2:

$$
\bar{a} = -18\, p\, q\, r s \frac{\xi_2(\xi_2 - 1)(2\xi_2 - 1)[p_+ + r_-(2\xi_2 - 1)]^2 [r_- + p_+(2\xi_2 - 1)]}{P(\xi_2)^2} \tag{5.8}
$$

$$
P(\xi_2) = 4p_+(r_-^2 - p_-^2)\,\xi_2^3 + 2\left[2r_-^3 - r_-(p_+^2 + p_-^2) - 3p_+(r_-^2 - p_-^2)\right]\ \xi_2^2 \tag{5.9}
$$

$$
+2(p_+ - r_-)(2r_-^2 - p_-^2 - p_+^2)\,\xi_2 + (p_+ + r_-)(p_+ - r_-)^2\,. \tag{5.10}
$$

We can rewrite $\bar{a}$ in terms of a variable W as

$$
\bar{a} = \frac{1}{4}\frac{8pqrs}{(p+q)^3}\frac{1}{W} \tag{5.11}
$$

Since ξ_2 obeys the quartic equation (5.5) that implies that W also satisfies a similar equation. Using a computer algebra program (e.g. Maple or Mathematica), it is easy to check that the equation W satisfies is[7]:

$$
0 = (1 - f^2)(1 - g^2)h_-^4 + 2h_-^2\left[2\,(2 - h_+)^2 - 3h_-^2\right] W \tag{5.12}
$$

$$
\left[8h_+\,(2 - h_+)^2 - h_-^2\,(30 + 9h_+)\right] W^2 \tag{5.13}
$$

$$
+6(2 - 9h_+)W^3 - 27W^4 \tag{5.14}
$$

[6]This result is obtained by computing $\tilde{a} = a - \frac{1}{3}(x\partial_x a + y\partial_y a + z\partial_z a)$ which at the extremes agrees with a.

[7]To check this, one replaces (5.10) in this equation. After taking common denominator the numerator can be seen to factorize into $P_{[4]}(\xi_2)$ and a polynomial of order twenty. Since, by (5.5), $P_{[4]}(\xi_2)$=0, the equation is satisfied.

where $f = -p_-/p_+$, $g = r_-/p_+$ and $h_\pm = f^2 \pm g^2$. This equation is precisely the same equation that appeared in [15] and determines the volume $V = \pi^3(p+q)^3 W/(8pqrs)$ of the $L^{p,q|r}$ manifold. This implies that the AdS/CFT relation [1, 41]

$$a = \frac{\pi^3}{4\mathrm{V}} \tag{5.15}$$

between a and the volume V of the manifold is exactly satisfied. It is perhaps interesting that even if only one solution of the quartic equation for the volume is physical all solutions actually match. This suggest that it might be possible to do a more direct derivation of the equivalence between the supergravity and field theory computations.

5.3 BPS operators and massless geodesics

Now we would like to compare the flavor and R-charges of the operators that we associate with the geodesics at the "corners" of the geometry and that we summarized in table 1. The charges of the corresponding operators are summarized in table 3. One thing to note is that since $R_\swarrow = R_\downarrow + R_\leftharpoondown$ and $R_\nwarrow = R_\uparrow + R \leftarrow$ then the R-charges of these particular operators depend only on their total Q_H and Q_V charges. For other operators that is not the case, for example there are operators with large R-charge and $Q_H = Q_V = 0$, that arise taking powers of one basic operator $\mathcal{O}_\beta$ with $Q_R = 2$ and $Q_H = Q_V = 0$. This short BPS meson $\mathcal{O}_\beta$ generates the β-deformation [24, 25] and exists for any toric superconformal quiver [24].

Our first task is to relate the $U(1)$ flavor charges Q_V and Q_H with the isometries Q_ϕ and Q_ψ of the background. We found that we obtain a correct matching if we define

$$Q_V = \frac{1}{2}\left(Q_\psi - Q_\phi\right) \tag{5.16}$$

$$Q_H = -\frac{(AQ_\phi + BQ_\psi)}{2\,p\,q\,(x_2 - x_1)} \tag{5.17}$$

$$A = ps(\alpha - x_1) + qs(\alpha - x_2) \tag{5.18}$$

$$B = pr(\beta - x_1) + qr(\beta - x_2)\,. \tag{5.19}$$

The R-charges of the operators can be computed from eqn. (3.5) (see table 3), with the result

$$R_{LD} = -q\bar{y} + s(1 - \bar{x} + \bar{z}) \tag{5.20}$$

$$R_{RU} = r(1 + \bar{x}) + p(\bar{y} - \bar{z}) \tag{5.21}$$

$$R_{LU} = r(1 - \bar{x} + \bar{z}) + q(\bar{y} - \bar{z}) \tag{5.22}$$

$$R_{RD} = s(1 + \bar{x}) - p\bar{y}\,. \tag{5.23}$$

We remind the reader that $(\bar{x}, \bar{y}, \bar{z})$ indicates (x, y, z) evaluated at the local maximum of the central charge a. For the ratios Q_V/Q_R and Q_H/Q_R to match between field theory and supergravity background we need that

$$-\frac{3}{2}\left(1 - \frac{x_1}{\alpha}\right) = \frac{-q}{-q\bar{y} + s(1 - \bar{x} + \bar{z})} \tag{5.24}$$

$$\frac{3}{2}\left(1-\frac{x_2}{\beta}\right) = \frac{p}{r(1+\bar{x})+p(\bar{y}-\bar{z})} \tag{5.25}$$

$$\frac{3}{2}\left(1-\frac{x_1}{\beta}\right) = \frac{q}{r(1-\bar{x}+\bar{z})+q(\bar{y}-\bar{z})} \tag{5.26}$$

$$-\frac{3}{2}\left(1-\frac{x_2}{\alpha}\right) = \frac{-p}{-p\bar{y}+s(1+\bar{x})}\,. \tag{5.27}$$

In the appendix we show that these relations are exactly valid. In fact, together with a further relation

$$\frac{x_3}{\alpha} = -\frac{2}{3\bar{y}} \tag{5.28}$$

$$\frac{x_3}{\beta} = \frac{2}{3(\bar{y}-\bar{z})} \tag{5.29}$$

can be used to compute all parameters of the geometry in terms of field theory quantities. We emphasize that these simple relations were found thanks to the method of comparing massless geodesics with BPS operators.

6. Conclusions

In the present paper we consider massless geodesics moving in the recently found $L^{p,q|r}$ backgrounds. The study of the geodesics give considerable information about the field theory, in particular they determine a set of four operators which have maximal charges (in modulus) for given R-charge. These are operators which are constituted by elementary fields all with the same sign of each charge. We find four of them, in correspondence with the signs of the two flavor charges. On the other hand an analysis of the toric diagrams of the theories and comparison with the previously known $\mathcal{Y}^{p,q}$ case suggest a generic construction of the toric representation of the quiver. This allows us to conjecture the generic superconformal theories dual to the $L^{p,q|r}$ manifolds. For those theories we compute the R-charges using a-maximization and find that the result precisely matches the computation done in the geometry. In particular a precise mapping is found between the parameters of the geometry and those that arise in the field theory when performing a-maximization. The analysis is straight-forward albeit cumbersome. For that reason we choose an example of interest, the so called "generalized conifolds" which can be identified with $L^{p,q|q}$. In that case we compute explicitly all the R-charges. In the generic case the results are written in terms of the solutions of a quartic equation on both sides of the correspondence. The agreement is shown by verifying that the solutions on one side satisfy the equations on the other side of the correspondence.

In further work, it would be interesting to do a study of extended semiclassical strings [45] in these backgrounds, as was done in [22] for the $\mathcal{Y}^{p,q}$ case.

It would also be interesting to see if there is a way of finding the properties of geodesics starting directly from the toric diagrams. A similar understanding has been achieved for the volumes in [46].

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7. Useful formulas

7.1 The geometry

The manifold $L^{p,q|r}$ is defined in terms of two different sets of parameters. One is $(\alpha,\,\beta)$ and the roots $x_1 < x_2 < x_3$ of the cubic equation $x(\alpha - x)(\beta - x) = \mu$. The other are the integers p,q,r. Here we are interested in an explicit relation between the two sets that we derive following [15].

The roots $x_{1,2,3}$ satisfy (we assume from now on $\mu = 1$)

$$x_1 x_2 x_3 = 1, \quad x_1x_2 + x_1x_3 + x_2x_3 = \alpha\beta, \quad x_1 + x_2 + x_3 = \alpha + \beta\,. \tag{7.1}$$

From [15], the relation to the integers p,q,r is given through a set of parameters A_i, B_i , C_i, $i = 1, 2$ defined as

$$A_i = \frac{\alpha C_i}{x_i - \alpha}, \quad B_i = \frac{\beta C_i}{x_i - \beta}, \quad C_i = \frac{(\alpha - x_i)(\beta - x_i)}{2(\alpha + \beta)x_i - \alpha\beta - 3x_i^2} \tag{7.2}$$

and which satisfy

$$pC_1 + qC_2 = 0, \quad pA_1 + qA_2 + r = 0, \quad pB_1 + qB_2 + s = 0\,. \tag{7.3}$$

Using eq. (7.1) we can write

$$C_1 = -\frac{1}{x_1(x_1 - x_2)(x_1 - x_3)}, \quad C_2 = -\frac{1}{x_2(x_2 - x_1)(x_2 - x_3)}\,. \tag{7.4}$$

We can derive now two equations relating the x_i's to the integers p,q,r:

$$px_2(x_2 - x_3) = qx_1(x_1 - x_3), \quad x_1x_2 + x_1x_3 + x_2x_3 = \frac{rs}{pq}(x_1 - x_3)(x_2 - x_3) \tag{7.5}$$

which together with $x_1x_2x_3 = 1$ completely determine x_i in terms of p,q,r. To solve these equations we introduce the ratios

$$\chi_1 = \frac{x_1}{x_3}, \quad \chi_2 = \frac{x_2}{x_3}\,. \tag{7.6}$$

The equations now reduce to

$$p\chi_2(1 - \chi_2) = q\chi_1(1 - \chi_1), \qquad \chi_1\chi_2 + \chi_1 + \chi_2 = \frac{rs}{pq}(1 - \chi_1)(1 - \chi_2)\,. \tag{7.7}$$

The second equation allows to obtain χ_2 as:

$$\chi_2 = \frac{-p\,q\,\chi_1 + rs\,(1 - \chi_1)}{p\,q\,(1 + \chi_1)\; + \,rs\,(1 - \chi_1)}\,. \tag{7.8}$$

Replacing in the first one, we find a quartic equation for χ_1:

$$\begin{aligned} 0 = {} & (p\,q - rs)^2\chi_1^4 + (p\,q - rs)\,(3rs + p\,q)\chi_1^3 \\ & + (p\,q + rs)\,(3rs - 2p^2 - p\,q)\chi_1^2 + [p^2(rs - p\,q) - (p\,q + rs)^2]\chi_1 + p^2 rs \end{aligned} \tag{7.9}$$

The other parameters follow trivially as

$$x_1 = \left(\frac{\chi_1^2}{\chi_2}\right)^{\frac{1}{3}}, \quad x_2 = \left(\frac{\chi_2^2}{\chi_1}\right)^{\frac{1}{3}}, \quad x_3 = \frac{1}{(\chi_1\chi_2)^{\frac{1}{3}}} \tag{7.10}$$

$$\alpha = \frac{1+\chi_1+\chi_2+\sqrt{1-2\chi_1+\chi_1^2-2\chi_2+\chi_2^2-2\chi_1\chi_2}}{2(\chi_1\chi_2)^{\frac{1}{3}}} \tag{7.11}$$

$$\beta = \frac{1+\chi_1+\chi_2-\sqrt{1-2\chi_1+\chi_1^2-2\chi_2+\chi_2^2-2\chi_1\chi_2}}{2(\chi_1\chi_2)^{\frac{1}{3}}}. \tag{7.12}$$

We finally write

$$\frac{x_i}{\alpha} = \frac{2\chi_i}{1+\chi_1+\chi_2+\sqrt{1-2\chi_1+\chi_1^2-2\chi_2+\chi_2^2-2\chi_1\chi_2}} \tag{7.13}$$

$$\frac{x_i}{\beta} = \frac{2\chi_i}{1+\chi_1+\chi_2-\sqrt{1-2\chi_1+\chi_1^2-2\chi_2+\chi_2^2-2\chi_1\chi_2}} \tag{7.14}$$

from which it possible to find the value of the $U(1)$-fibration functions at the four vertices of the coordinate rectangle. Note that, although we set $\mu = 1$, the results for any ratio of two of the quantities x_i, α, β is independent of μ.

7.2 Map to the field theory

Now we want to find the relation between the parameters $x_{i=1\ldots3}$, α, β in the geometry and those in the field theory. The parameters we consider in the field theory are x, y, z used in section 5, when performing a-maximization. Here we consider them always evaluated at the local maximum in which case they are functions of p, q and r as determined in that section. To emphasize that they are evaluated at the local maximum we denote them as $\bar{x}$, $\bar{y}$ and $\bar{z}$.

Analyzing the matching to massless geodesics we were led to certain relations that can be summarized as follows:

$$\zeta_1 = \frac{x_1}{\alpha} = 1 + \frac{2}{3}\frac{q}{q\bar{y}-s(1-\bar{x}+\bar{z})} \tag{7.15}$$

$$\zeta_2 = \frac{x_2}{\alpha} = 1 - \frac{2}{3}\frac{p}{-p\bar{y}+s(1+\bar{x})} \tag{7.16}$$

$$\zeta_3 = \frac{x_3}{\alpha} = -\frac{2}{3}\frac{1}{\bar{y}} \tag{7.17}$$

$$\tilde{\zeta}_1 = \frac{x_1}{\beta} = 1 - \frac{2}{3}\frac{q}{q(\bar{y}-\bar{z})+r(1-\bar{x}+\bar{z})} \tag{7.18}$$

$$\tilde{\zeta}_2 = \frac{x_2}{\beta} = 1 - \frac{2}{3}\frac{p}{p(\bar{y}-\bar{z})+r(1+\bar{x})} \tag{7.19}$$

$$\tilde{\zeta}_3 = \frac{x_3}{\beta} = \frac{2}{3}\frac{1}{\bar{y}-\bar{z}}. \tag{7.20}$$

It is easier, as we did, to write these relations in terms of ratios. In the geometry this amounts to eliminating the parameter μ. To prove these relations, we proceed to show that they satisfy the same equations that we derived in the previous subsection (after appropriately dividing by α and eliminating β/α:

$$p\,\zeta_2\,(\zeta_2-\zeta_3)=q\,\zeta_1\,(\zeta_1-\zeta_3) \tag{7.21}$$

$$\zeta_1\zeta_2+\zeta_1\zeta_3+\zeta_2\zeta_3=\frac{rs}{p\,q}(\zeta_1-\zeta_3)(\zeta_2-\zeta_3) \tag{7.22}$$

$$\zeta_1+\zeta_2+\zeta_3=1+\zeta_1\zeta_2+\zeta_1\zeta_3+\zeta_2\zeta_3\,. \tag{7.23}$$

To check these equations first one replaces x, y and z by their expressions in terms of ξ_2, namely following eqs. (5.2), (5.4). After that, the equation becomes a rational function whose numerator is a polynomial multiple of $P_{[4]}(\xi_2)$ as defined in (5.6). Since, at the local extrema, ξ_2 is a root of $P_{[4]}(\xi_2)$, the equations are satisfied.

These equation completely determine $\zeta_{i=1\ldots3}$ up to a discrete set of permutations. We checked using particular examples that the assignments are as we discussed in the case $r>s$ that we are considering.

The same applies to the $\tilde{\zeta}_{i=1\ldots3}$. Moreover, as an exercise, one can check that other relations such as $\zeta_1\tilde{\zeta}_2-\tilde{\zeta}_1\zeta_2=0$ also reduce to zero after using that $P_{[4]}(\xi_2)=0$.

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